On a group-theoretical generalization of the Euler's totient function

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Abstract

Let G be a finite group and $\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$, where $o(a)$ denotes the order of a in G and $\exp(G)$ denotes the exponent of G. Under a natural hypothesis, in this note we determine the groups G such that $\forall\, H, K \leq G$, $H \subseteq K$ implies $\varphi(H) \mid \varphi(K)$. This partially answers Problem 5.4 in [6].

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1 Introduction

Given a finite group G, we consider the function

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$$

introduced in our previous paper [6]. Since $\varphi(\mathbb{Z}_n) = \varphi(n)$, for all $n \in \mathbb{N}^*$, it generalizes the well-known Euler's totient function. In [7], the function φ was used to provide a nilpotency criterion for finite groups, namely

$$G \text{ is nilpotent if and only if } \varphi(S) \neq 0 \text{ for any section } S \text{ of } G.$$

This has been extended to a p-nilpotency criterion by A. Díaz Ramos and A. Viruel [3]. We recall that the weaker condition

$$\varphi(S) \neq 0 \text{ for any subgroup } S \text{ of } G \tag{1}$$

does not guarantee the nilpotency of G, as it is shown in [7].

In the current note, we will describe finite groups G such that $\forall\, H, K \leq G$, $H \subseteq K$ implies $\varphi(H) \mid \varphi(K)$. To avoid the case in which $\varphi(H) = 0$, we will assume the condition (1) to be satisfied. Thus, we partially answer Problem 5.4 in [6].

Our main result is stated as follows.

Theorem 1.1. *Let G be a finite group satisfying the condition (1). Then*

$$\forall\, H, K \leq G, H \subseteq K \text{ implies } \varphi(H) \mid \varphi(K) \tag{2}$$

if and only if G is nilpotent and its Sylow subgroups are cyclic, Q_8 or $\mathbb{Z}_p \times \mathbb{Z}_p$ with p prime.

Next we recall two important classes of finite groups that will be used in our note:

- A *generalized quaternion* 2*-group* is a group of order 2^n for some positive integer $n \geq 3$, defined by the presentation

  $$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1} \rangle.$$

  These groups are the unique finite non-cyclic p-groups all of whose abelian subgroups are cyclic, or equivalently the unique finite non-cyclic p-groups possessing exactly one subgroup of order p (see e.g. (4.4) of [5], II).

- A *Schmidt group* is a non-nilpotent group all of whose proper subgroups are nilpotent. By [4] (see also [1,2]), such a group G is solvable of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p-subgroup P and a cyclic Sylow q-subgroup Q, and hence G is a semidirect product of P by Q. Moreover, we have:

  - if $Q = \langle y \rangle$, then $y^q \in Z(G)$;
  - $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P$, $P' = (G')' = \Phi(P)$;
  - $|P/P'| = p^r$, where r is the order of p modulo q;
  - if P is abelian, then P is an elementary abelian p-group of order p^r and P is a minimal normal subgroup of G;
  - if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

> We mention that $G/Z(G)$ is also a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian p-group of order p^r by a cyclic group of order q, and that it does not have cyclic subgroups of order pq.

Most of our notation is standard and will not be repeated here. Basic definitions and results on group theory can be found in [5].

2 Proof of Theorem 1.1

First of all, we prove two auxiliary results.

Lemma 2.1. *Let G be a finite p-group satisfying the condition (2). Then G is cyclic, Q_8 or $\mathbb{Z}_p \times \mathbb{Z}_p$.*

Proof. Assume first that G is abelian. If it is not cyclic, then there exists $H \leq G$ such that $H \cong \mathbb{Z}_p^2$. Let K be a subgroup of order p^3 of G which contains H. Then either $K \cong \mathbb{Z}_p^3$ or $K \cong \mathbb{Z}_p \times \mathbb{Z}_p^2$. It follows that $\varphi(H) = p^2-1$ divides either $\varphi(\mathbb{Z}_p^3) = p^3-1$ or $\varphi(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) = p^2(p-1)$, a contradiction. Thus such a subgroup K does not exist, showing that either G is cyclic or $G = H \cong \mathbb{Z}_p^2$.

Assume now that G is not abelian. Since the condition (2) is inherited by subgroups, we infer that all abelian subgroups of G are either cyclic or of type $\mathbb{Z}_p^2$. Suppose that G has an abelian subgroup $H \cong \mathbb{Z}_p^2$. Then H is contained in a subgroup K of order p^3 of one of the following types:

- $\mathbb{Z}_p^3$;
- $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$;
- $M(p^3) = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1} \rangle$;
- $E(p^3) = \langle x, y \mid x^p = y^p = [x, y]^p = 1, [x, y] \in Z(E(p^3)) \rangle$;
- D_8.

It is easy to see that in all cases $\varphi(H)$ does not divide $\varphi(K)$, contradicting our hypothesis. Consequently, all abelian subgroups of G are cyclic, implying that $G \cong Q_{2^n}$ for some $n \geq 3$. If $n \geq 4$, then G has a subgroup of type Q_8 and so

$$6 = \varphi(Q_8) \mid \varphi(G) = 2^{n-2},$$

a contradiction. Hence $G \cong Q_8$, as desired. $\square$

Lemma 2.2. *Let G be a finite group satisfying the conditions (1) and (2). Then G is nilpotent.*

Proof. Assume that G is a counterexample of minimal order. Then all proper subgroups of G also satisfy the conditions (1) and (2), and therefore G is a Schmidt group. Suppose that it has the structure described in Introduction. By Lemma 2.1, we distinguish the following three cases:

a) P is cyclic

Then $\exp(G) = p^m q^n$ and so the condition $\varphi(G) \neq 0$ implies that G is cyclic, a contradiction.

b) $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ with p prime

Then $Z(G) = \langle y^q \rangle$ and $\exp(G) = pq^n$. Since $\varphi(G) \neq 0$, there exists a cyclic subgroup $M \leq G$ of order pq^n. Note that we have $Z(G) \subset M$. It follows that the Schmidt group $G/Z(G)$ of order $p^2 q$ contains the cyclic subgroup $M/Z(G)$ of order pq, a contradiction.

c) $P \cong Q_8$

We have $|\mathrm{Aut}(Q_8)| = 24$. Since G is a non-trivial semidirect product of Q_8 and $\mathbb{Z}_{q^n}$, we get $q = 3$ and so $G \cong Q_8 \rtimes \mathbb{Z}_{3^n}$. Similarly with b), we obtain that the Schmidt group $G/Z(G) \cong A_4$ has a cyclic subgroup of order 6, a contradiction.

Hence such a group G does not exist, as desired. □

We are now able to prove our main result.

Proof of Theorem 1.1. The direct implication follows from Lemmas 2.1 and 2.2. Conversely, we observe that it suffices to prove the condition (2) for cyclic p-groups, Q_8 or $\mathbb{Z}_p \times \mathbb{Z}_p$ with p prime because a nilpotent group is the direct product of its Sylow subgroups. This is obvious, completing the proof. □

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